## Elementary mathematical expressions

### Quadratic equations

$$a^2 - b^2 = (a+b)(a-b) \tag{1}$$

The solutions to the general quadratic equation

$$ax^2 + bx + c = 0 \tag{2}$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

### Taylor and Maclaurin series (power-series expansion)

The Taylor series

$$f(x) = f(a) + \frac{(x-a)}{1!}\frac{df}{dx}\bigg|_a + \frac{(x-a)^2}{2!}\frac{d^2f}{dx^2}\bigg|_a + \ldots + \frac{(x-a)^n}{n!}\frac{d^nf}{dx^n}\bigg|_a + \ldots \tag{4}$$

gives a useful way of approximating a function near to some specific point $x = a$, giving a power-series expansion in $(x-a)^n$ for the function near that point.

The Maclaurin series

$$f(x) = f(0) + \frac{x}{1!}\frac{df}{dx}\bigg|_0 + \frac{x^2}{2!}\frac{d^2f}{dx^2}\bigg|_0 + \ldots + \frac{x^n}{n!}\frac{d^nf}{dx^n}\bigg|_0 + \ldots \tag{5}$$

is a special case of the Taylor series where we are expanding around the point $x = 0$.

### Power-series expansions of common functions

For small $a$, the Maclaurin expansions of various common functions are, to first order

$$\sqrt{1+a} \simeq 1 + a/2 + \ldots \tag{6}$$

$$\frac{1}{1+a} \simeq 1 - a + \ldots \tag{7}$$

$$\sin a \simeq a + \ldots \tag{8}$$

$$\tan a \simeq a + \ldots \tag{9}$$

$$\cos a \simeq 1 - \frac{a^2}{2} + \ldots \tag{10}$$

$$\exp a \simeq 1 + a + \ldots \tag{11}$$

### Sine and cosine addition and product formulae

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{12}$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{13}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \tag{14}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{15}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) \tag{16}$$

$$\cos^2(\alpha) = \frac{1}{2}\left[1 + \cos(2\alpha)\right] \tag{17}$$

$$\sin^2(\alpha) = \frac{1}{2}\left[1 - \cos(2\alpha)\right] \tag{18}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right] \tag{19}$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right] \tag{20}$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right] \tag{21}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{22}$$

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{23}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{24}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{25}$$

## Differential calculus

### Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{26}$$

### Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{27}$$

### Chain rule

$$\frac{d}{dx}f(g(x)) = \left(\frac{df}{dg}\right) \times \left(\frac{dg}{fx}\right) \tag{28}$$

### Derivatives of elementary functions

$$\frac{d}{dx}x^n = nx^{n-1} \tag{29}$$

$$\frac{d}{dx}\exp(ax) = a\exp(ax) \tag{30}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \tag{31}$$

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{32}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{33}$$

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}} \tag{34}$$

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2} \tag{35}$$

## Integral calculus

### Integration by parts

$$\int_a^b f(x)\left(\frac{dg(x)}{dx}\right)dx = \left[f(x)g(x)\right]_a^b - \int_a^b \left(\frac{df(x)}{dx}\right)g(x)\,dx \tag{36}$$

where we use the common notation

$$\left[h(x)\right]_a^b = h(b) - h(a) \tag{37}$$

and, specifically, here

$$\left[f(x)g(x)\right]_a^b = f(b)g(b) - f(a)g(a) \tag{38}$$

### Some definite integrals

$$\int_0^\pi \sin^2(nx)\,dx = \frac{\pi}{2} \tag{39}$$

$$\int_0^\pi (x - \pi/2)\sin(nx)\sin(mx)\,dx = \frac{-4nm}{(n-m)^2(n+m)^2}\,, \quad \text{for } n+m \text{ odd} \tag{40}$$
$$= 0\,, \quad \text{for } n+m \text{ even}$$

$$\int_0^\pi \sin(\theta)\cos(2\theta)\,d\theta = -2/3 \tag{41}$$

$$\int_0^\pi \sin(2\theta)\cos(\theta)\,d\theta = 4/3 \tag{42}$$

$$\int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3} \tag{43}$$

$$\int_0^\infty t^{1/2}\exp(-t)\,dt = \frac{\sqrt{\pi}}{2} \tag{44}$$

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi \tag{45}$$

$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi \tag{46}$$

$$\int_{-\infty}^\infty \exp(-x^2)\,dx = \sqrt{\pi} \tag{47}$$

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi \tag{48}$$

## Partial differentiation

For a function $h(x, y)$ that is a function of two independent variables $x$ and $y$, the partial derivative, often stated as "partial d h by d x" or, more explicitly, "partial d h by d x at constant y", and written as

$$\frac{\partial h}{\partial x} \equiv \frac{\partial h}{\partial x}\bigg|_{y} \tag{49}$$

is the derivative of $h$ with respect to $x$ with the $y$ variable held at a constant value. That value can also be explicitly stated, for example, as in the notation

$$\frac{\partial h}{\partial x}\bigg|_{y=y_o} \tag{50}$$

which would be the partial derivative taken at the specific value $y = y_o$.

Higher partial derivatives can be formed similarly, as in the notations

$$\frac{\partial^2 h}{\partial x^2} \equiv \frac{\partial^2 h}{\partial x^2}\bigg|_{y} \tag{51}$$

and, for the "cross derivative",

$$\frac{\partial^2 h}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\bigg|_{y} \frac{\partial h}{\partial y}\bigg|_{x} \tag{52}$$

Provided all the various first derivatives and the two cross-derivatives in the two different orders both exist, we can interchange the order of the partial differentiations in the cross-derivative; that is,

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \tag{53}$$

For small or infinitesimal changes $dx$ in $x$ and $dy$ in $y$, the resulting total change in $h$ or *differential* or *exact differential* is written

$$dh = \frac{\partial h}{\partial x}\bigg|_{y} dx + \frac{\partial h}{\partial y}\bigg|_{x} dy \tag{54}$$

If $x$ and $y$ are both functions of some other variable $t$, then the *total derivative $dh/dt$* is given by

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\bigg|_{y}\left(\frac{dx}{dt}\right) + \frac{\partial h}{\partial y}\bigg|_{x}\left(\frac{dy}{dt}\right) \tag{55}$$

If $x$ and $y$ are each themselves functions of two variables $a$ and $b$, then we can write

$$\frac{\partial h}{\partial a}\bigg|_{b} = \frac{\partial h}{\partial x}\bigg|_{y} \frac{\partial x}{\partial a}\bigg|_{b} + \frac{\partial h}{\partial y}\bigg|_{x} \frac{\partial y}{\partial a}\bigg|_{b} \tag{56}$$

Because this works for any function of $x$ and $y$ (for which all appropriate derivative exist), we can write

$$\left.\frac{\partial}{\partial a}\right|_{b} = \left.\frac{\partial x}{\partial a}\right|_{b} \left.\frac{\partial}{\partial x}\right|_{y} + \left.\frac{\partial y}{\partial a}\right|_{b} \left.\frac{\partial}{\partial y}\right|_{x} \tag{57}$$

which can be used to change partial derivatives from one coordinate system to another.

## Vector calculus

### *Cartesian coordinates*

The the $\nabla$ operator, which occurs in various different vector calculus operators, is known as *del* or *nabla*, can be written as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{58}$$

in Cartesian coordinates, with $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ as unit vectors in the $x$, $y$, and $z$ directions respectively.

The *gradient* operator operates on a scalar function $f(x, y, z)$ to give a vector whose magnitude and direction are the slope or gradient of the scalar function at the point of interest. In Cartesian coordinates

$$\operatorname{grad} f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \tag{59}$$

The *Laplacian* operator, also known as *del squared*, operates on a scalar function, giving a scalar result. It is written in Cartesian coordinates as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{60}$$

The operator $\nabla \cdot \nabla$, sometimes also written as $\nabla^2$, can operate on a vector function, in which case, in Cartesian coordinates, we have

$$\left(\nabla \cdot \nabla\right)\mathbf{F} = \mathbf{i}\frac{\partial^2 F_x}{\partial x^2} + \mathbf{j}\frac{\partial^2 F_y}{\partial y^2} + \mathbf{k}\frac{\partial^2 F_z}{\partial z^2} \tag{61}$$

In Cartesian coordinates, the *divergence* of a vector $\mathbf{F}$ is defined as

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{62}$$

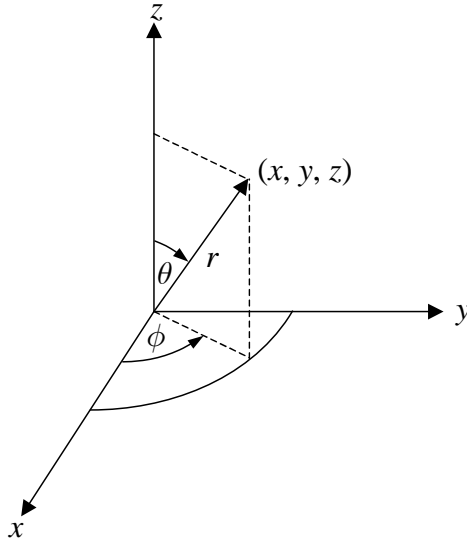In Cartesian coordinates, the *curl* of a vector $\mathbf{F}$ is defined as

$$\operatorname{curl}\mathbf{F} \equiv \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \tag{63}$$

or in the equivalent "determinant" shorthand form,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{64}$$

### Spherical polar coordinates

In spherical polar coordinates, which can be defined as in the following diagram



with

$$x = r\sin\theta\cos\phi \tag{65}$$

$$y = r\sin\theta\sin\phi \tag{66}$$

$$z = r\cos\phi$$

the *gradient* can be written

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} \tag{67}$$

the *Laplacian* can be written

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{68}$$

the *divergence* can be written

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(F_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{69}$$

and the *curl* can be written

$$\nabla\times\mathbf{F} = \begin{vmatrix} \dfrac{\hat{\mathbf{r}}}{r^2\sin\theta} & \dfrac{\hat{\theta}}{r\sin\theta} & \dfrac{\hat{\phi}}{r} \\[2mm] \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\[2mm] F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \tag{70}$$

### *Vector calculus identities*

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{71}$$

$$\nabla \times \nabla f = 0 \tag{72}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla) \mathbf{F} \tag{73}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = -\mathbf{F} \cdot (\nabla \times \mathbf{G}) + \mathbf{G} \cdot (\nabla \times \mathbf{F}) \tag{74}$$

$$\Psi \nabla^2 \Psi^* - \Psi^* \nabla^2 \Psi = \Psi \nabla^2 \Psi^* + \nabla \Psi \nabla \Psi^* - \nabla \Psi \nabla \Psi^* - \Psi^* \nabla^2 \Psi$$
$$= \nabla \cdot (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \tag{75}$$